Unified entropy, entanglement measures and monogamy of multi-party entanglement

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Abstract. We show that restricted shareability of multi-qubit entanglement can be fully characterized by unified-(q, s) entropy. We provide a two-parameter class of bipartite entanglement measures, namely unified-(q, s) entanglement with its analytic formula in two-qubit systems for $q \geq 1$, $0 \leq s \leq 1$ and $qs \leq 3$. Using unified-(q, s) entanglement, we establish a broad class of the monogamy inequalities of multi-qubit entanglement for $q \geq 2$, $0 \leq s \leq 1$ and $qs \leq 3$.

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1. Introduction

Quantum entanglement is a physical resource with various applications to quantum information and communication processing. Quantum teleportation uses maximal entanglement between two particles as a resource to transfer an unknown quantum state from one particle to another without sending the actual particle itself [1]. The non-local correlation of quantum entanglement also provides us with secure cryptographic keys [2, 3].

Whereas classical correlation can be freely shared among parties in multi-party systems, quantum entanglement is restricted in its shareability. If a pair of parties are maximally entangled in multipartite systems, they cannot share entanglement [4, 5] nor classical correlations [6] with the rest of the system, thus the term *monogamy of entanglement* (MoE) [7].

MoE lies at the heart of many quantum information and communication protocols. In quantum cryptography, for example, MoE is fundamentally important because it quantifies how much information an eavesdropper could potentially obtain about the secret key to be extracted; the founding principle of quantum cryptographic schemes that an eavesdropper cannot obtain any information without disturbance is guaranteed the law of quantum physics, namely MoE rather than assumptions on the difficulty of computation.

The first mathematical characterization of MoE was established for three-qubit systems as an inequality in terms of concurrence [8], which is referred as the Coffman-Kundu-Wootters (CKW) inequality [4]. The CKW inequality was generalized for multi-qubit systems [5] and for some cases of multi-qudit systems [9], and dual monogamy inequalities were also proposed for multi-party quantum systems [10, 11, 12].

However, there exist quantum states in higher-dimensional systems violating CKW inequality [13, 14]; thus the CKW inequality in multi-qubit systems fails in its generalization into higher-dimensional quantum systems. Moreover, characterizing MoE as an inequality is not generally true for other entanglement measures such as *entanglement of formation* (EoF) [15]; monogamy inequality in terms of EoF is not valid even in multi-qubit systems. Thus it is important to have a proper entanglement measures to characterize MoE not only for the study of general MoE in higher-dimensional quantum systems but in multi-qubit systems as well.

The proof of the CKW inequality in multi-qubit systems [4] is based on the feasibility of analytic evaluation of concurrence for two-qubit mixed states. In fact, there are various possible definitions of bipartite entanglement measure using different entropy functions such as Rényi-α and Tsallis-q entropies [16, 17, 18, 19]. For selective ranges of α and q, these entanglement measures are tractable in two-qubit systems, and, moreover, monogamy inequality of multi-qubit entanglement is feasible in terms of these measures [22, 23].

Here we establish a unification of monogamy inequalities in multi-qubit systems. Using unified-(q, s) entropy with real parameters q and s [20, 21], we define a class

of bipartite entanglement measures namely *unified-(q,s) entanglement*, and show a broad class of monogamy inequalities of multi-qubit systems in terms of unified-(q,s) entanglement.

Our result shows that unified-(q,s) entanglement contains concurrence, EoF, Rényi-α and Tsallis-q entanglement as special cases, showing their explicit relation with respect to a smooth function. Furthermore, our result reduces to every known case of multi-qubit monogamy inequalities such as Rényi and Tsallis monogamy [22, 23] and the CKW inequality for selective choices of q and s. Thus, our result provides an interpolation of the previous results about monogamy of multi-qubit entanglement.

This paper is organized as follows. In Section 2.1, we define unified-(q,s) entanglement for bipartite quantum states, and provide its relation with concurrence, EoF, Rényi-q entanglement and Tsallis-q entanglement. In Section 2.2, we provide an analytic formula of unified-(q,s) entanglement in two-qubit systems for $q \geq 1$, $0 \leq s \leq 1$ and $qs \leq 3$. In Section 3, we derive a monogamy inequality of multi-qubit entanglement in terms of unified-(q,s) entanglement for $q \geq 2$, $0 \leq s \leq 1$ and $qs \leq 3$. We summarize our results in Section 4.

2. Unified-(q,s) Entanglement

2.1. Definition

For a quantum state ρ, unified-(q,s) entropy is

$$S_{q,s}(\rho) := \frac{1}{(1-q)s}\left[(\mathrm{tr}\rho^q)^s - 1\right], \tag{1}$$

for $q,\ s \geq 0$ such that $q \neq 1$ and $s \neq 0$. Unified-(q,s) entropy converges to Rényi-q entropy [17],

$$\lim_{s\to 0} S_{q,s}(\rho) = \frac{1}{1-q}\log \mathrm{tr}\rho^q = R_q(\rho), \tag{2}$$

and also tends to Tsallis-q entropy [19],

$$\lim_{s\to 1} S_{q,s}(\rho) = \frac{1}{1-q}\left(\mathrm{tr}\rho^q - 1\right) = T_q(\rho). \tag{3}$$

For the case that q tends to 1, $S_{q,s}(\rho)$ converges to the von Neumann entropy, that is

$$\lim_{q\to 1} S_{q,s}(\rho) = -\mathrm{tr}\rho\log\rho = S(\rho). \tag{4}$$

Although unified-(q,s) entropy is singular for $q=1$ or $s=0$, we can consider them to be von Neumann entropy or Rényi-q entropy, respectively. For this reason, we let $S_{1,s}(\rho) \equiv S(\rho)$ and $S_{q,0}(\rho) \equiv R_q(\rho)$ for any quantum state ρ.

For a bipartite pure state $|\psi\rangle_{AB}$ and each $q,\ s \geq 0$, unified-(q,s) entanglement is

$$E_{q,s}\left(|\psi\rangle_{AB}\right) := S_{q,s}(\rho_A), \tag{5}$$

where $\rho_A = \mathrm{tr}_B|\psi\rangle_{AB}\langle\psi|$ is the reduced density matrix for subsystem A. For a mixed state ρ_{AB}, we define its unified-(q,s) entanglement via the convex-roof extension,

$$E_{q,s}\left(\rho_{AB}\right) := \min\sum_i p_i E_{q,s}(|\psi_i\rangle_{AB}), \tag{6}$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i|\psi_i\rangle_{AB}\langle\psi_i|$.

Because unified-(q,s) entropy converges to Rényi and Tsallis entropies when s tends to 0 and 1 respectively,

$$\lim_{s\to 0} E_{q,s}(\rho_{AB}) = \mathcal{R}_q(\rho_{AB}), \tag{7}$$

where $\mathcal{R}_q(\rho_{AB})$ is the Rényi-q entanglement of ρ_{AB} [22], and

$$\lim_{s\to 1} E_{q,s}(\rho_{AB}) = \mathcal{T}_q(\rho_{AB}), \tag{8}$$

where $\mathcal{T}_q(\rho_{AB})$ is the Tsallis-q entanglement [23]. For q tends to 1,

$$\lim_{q\to 1} E_{q,s}(\rho_{AB}) = E_{\mathrm{f}}(\rho_{AB}), \tag{9}$$

where $E_{\mathrm{f}}(\rho_{AB})$ is the EoF of ρ_{AB}. Thus unified-(q,s) entanglement is a two-parameter generalization of EoF.

2.2. Analytic formula of unified-(q,s) entanglement for two-qubit states

Let us recall concurrence and its functional relation with EoF in two-qubit systems. For any bipartite pure state $|\psi\rangle_{AB}$, its concurrence, $\mathcal{C}(|\psi\rangle_{AB})$ is

$$\mathcal{C}(|\psi\rangle_{AB}) = \sqrt{2(1-\mathrm{tr}\rho_A^2)}, \tag{10}$$

where $\rho_A = \mathrm{tr}_B(|\psi\rangle_{AB}\langle\psi|)$ [8]. For a mixed state ρ_{AB}, its concurrence is

$$\mathcal{C}(\rho_{AB}) = \min\sum_k p_k \mathcal{C}(|\psi_k\rangle_{AB}), \tag{11}$$

where the minimum is taken over all possible pure state decompositions, $\rho_{AB} = \sum_k p_k|\psi_k\rangle_{AB}\langle\psi_k|$.

For a two-qubit pure state $|\psi\rangle_{AB}$ with Schmidt decomposition

$$|\psi\rangle_{AB} = \sqrt{\lambda_0}|00\rangle_{AB} + \sqrt{\lambda_1}|11\rangle_{AB}, \tag{12}$$

its reduced density operator of subsystem A is

$$\rho_A = \mathrm{tr}_B(|\psi\rangle_{AB}\langle\psi|) = \lambda_0|0\rangle_A\langle 0| + \lambda_1|1\rangle_A\langle 1|. \tag{13}$$

From Eq. (10), we obtain

$$\mathcal{C}(|\psi\rangle_{AB}) = \sqrt{2(1-\mathrm{tr}\rho_A^2)} = 2\sqrt{\lambda_0\lambda_1}, \tag{14}$$

and, moreover,

$$2\sqrt{\lambda_0\lambda_1} = \left(\mathrm{tr}\sqrt{\rho_A}\right)^2 - 1 = S_{\frac{1}{2},2}(\rho_A) = E_{\frac{1}{2},2}(|\psi\rangle_{AB}), \tag{15}$$

where $E_{\frac{1}{2},2}(|\psi\rangle_{AB})$ is the unified-$(1/2,2)$ entanglement of $|\psi\rangle_{AB}$. In other words, unified-(q,s) entanglement of a two-qubit pure state $|\psi\rangle_{AB}$ coincides with its concurrence for $q = 1/2$ and $s = 2$. As both concurrence and unified-(q,s) entanglement of bipartite mixed states are defined via the convex-roof extension, we note that unified-(q,s)

entanglement of a two-qubit mixed state reduces to its concurrence when $q = 1/2$ and $s = 2$;

$$\mathcal{C}(\rho_{AB}) = E_{\frac{1}{2},2}\left(\rho_{AB}\right), \tag{16}$$

for a two-qubit state ρ_{AB}.

Concurrence has an analytic formula in two-qubit systems [8]. For a two-qubit state ρ_{AB},

$$\mathcal{C}(\rho_{AB}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{17}$$

where λ_i's are the eigenvalues, in decreasing order, of $\sqrt{\sqrt{\rho_{AB}}\tilde{\rho}_{AB}\sqrt{\rho_{AB}}}$ and $\tilde{\rho}_{AB} = \sigma_y \otimes \sigma_y \rho^*_{AB} \sigma_y \otimes \sigma_y$ with the Pauli operator σ_y. Furthermore, the relation between concurrence and EoF of a two-qubit mixed state ρ_{AB} (or a pure state $|\psi\rangle_{AB} \in \mathbb{C}^2 \otimes \mathbb{C}^d$, $d \geq 2$) is given as a monotonically increasing, convex function such that

$$E_{\rm f}(\rho_{AB}) = \mathcal{E}(\mathcal{C}\left(\rho_{AB}\right)), \tag{18}$$

where

$$\mathcal{E}(x) = H\left(\frac{1 - \sqrt{1 - x^2}}{2}\right), \quad \text{for } 0 \leq x \leq 1, \tag{19}$$

with the binary entropy function $H(t) = -[t\log t + (1-t)\log(1-t)]$ [8]. The functional relation between concurrence and EoF as well as the analytic formula of concurrence for two-qubit states provide an analytic formula of EoF in two-qubit systems.

Now let us consider the functional relation between unified-(q, s) entanglement and concurrence for two-qubit states. For any $2 \otimes d$ pure state $|\psi\rangle_{AB}$ with its Schmidt decomposition $|\psi\rangle_{AB} = \sqrt{\lambda}|00\rangle_{AB} + \sqrt{1-\lambda}|11\rangle_{AB}$, its unified-$(q, s)$ entanglement is

$$E_{q,s}\left(|\psi\rangle_{AB}\right) = S_{q,s}(\rho_A) = \frac{1}{(1-q)s}\left[\left(\lambda^q + (1-\lambda)^q\right)^s - 1\right]. \tag{20}$$

Because the concurrence of $|\psi\rangle_{AB}$ is

$$\mathcal{C}(|\psi\rangle_{AB}) = \sqrt{2(1 - \mathrm{tr}\rho_A^2)} = 2\sqrt{\lambda\left(1-\lambda\right)}, \tag{21}$$

we have

$$E_{q,s}\left(|\psi\rangle_{AB}\right) = f_{q,s}\left(\mathcal{C}(|\psi\rangle_{AB})\right), \tag{22}$$

where $f_{q,s}(x)$ is a differential function

$$f_{q,s}(x) := \frac{\left(\left(1 + \sqrt{1-x^2}\right)^q + \left(1 - \sqrt{1-x^2}\right)^q\right)^s - 2^{qs}}{(1-q)s2^{qs}} \tag{23}$$

on $0 \leq x \leq 1$. Thus for any $2 \otimes d$ pure state $|\psi\rangle_{AB}$, we have a functional relation between its concurrence and unified-(q, s) entanglement for each q and s.

We note that $f_{1/2,2}(x) = x$ is the identity function, which also reveals the coincidence of concurrence and unified-$(1/2, 2)$ entanglement in Eq. (16). Furthermore, $f_{q,s}(x)$ converges to $\mathcal{E}(x)$ in Eq. (19) as q tends to 1, and it reduces to functions that relate concurrence with Rényi-q entanglement and Tsallis-q entanglement as s tends to 0 and 1 respectively [22, 23]. For two-qubit mixed states, we have the following theorem.

Theorem 1. *For $q \geq 1$, $0 \leq s \leq 1$, $qs \leq 3$ and any two-qubit state ρ_{AB},*

$$E_{q,s}\left(\rho_{AB}\right) = f_{q,s}\left(\mathcal{C}(\rho_{AB})\right). \tag{24}$$

We note that Theorem 1 together with the analytic formula of two-qubit concurrence in Eq. (17) provide us with an analytic formula of unified-(q,s) entanglement in two-qubit systems. Before we prove Theorem 1, we have the following lemma.

Lemma 2. *For $q \geq 1$, $0 \leq s \leq 1$ and $qs \leq 3$, $f_{q,s}(x)$ is a monotonically-increasing convex function on $0 \leq x \leq 1$.*

Proof. Because $f_{q,s}(x)$ is a differentiable function on $0 \leq x \leq 1$, its monotonicity and convexity follow from nonnegativity of its first and second derivatives. Furthermore, for $q > 1$, the monotonicity and convexity of $f_{q,s}$ follows from those of a function,

$$g_{q,s}(x) := -\left[\left(1+\sqrt{1-x^2}\right)^q + \left(1-\sqrt{1-x^2}\right)^q\right]^s. \tag{25}$$

For

$$\Theta = 1+\sqrt{1-x^2},\ \Xi = 1-\sqrt{1-x^2}, \tag{26}$$

the first derivative of $g_{q,s}(x)$ is

$$\frac{\mathrm{d}g_{q,s}(x)}{\mathrm{d}x} = \frac{qsx}{\sqrt{1-x^2}}\left(\Theta^q+\Xi^q\right)^{s-1}\left(\Theta^{q-1}-\Xi^{q-1}\right), \tag{27}$$

which is always nonnegative on $0 \leq x \leq 1$ for $q \geq 1$. For the second derivative of $g_{q,s}(x)$, we have

$$\begin{aligned}\frac{\mathrm{d}^2g_{q,s}(x)}{\mathrm{d}x^2} =& \Lambda\frac{\left(\Theta^q+\Xi^q\right)\left(\Theta^{q-1}-\Xi^{q-1}\right)}{\sqrt{1-x^2}}\\ &+\Lambda q(1-s)x^2\left(\Theta^{q-1}-\Xi^{q-1}\right)^2\\ &-\Lambda(q-1)x^2\left(\Theta^q+\Xi^q\right)\left(\Theta^{q-2}+\Xi^{q-2}\right)\end{aligned} \tag{28}$$

with $\Lambda = qs\left(\Theta^q+\Xi^q\right)^{s-2}/(1-x^2)$.

Because $q(1-s) \geq q-3$ for $qs \leq 3$, we have

$$\begin{aligned}\frac{\mathrm{d}^2g_{q,s}(x)}{\mathrm{d}x^2} \geq& \Lambda\left(\Theta^q+\Xi^q\right)\left[\frac{\left(\Theta^{q-1}-\Xi^{q-1}\right)}{\sqrt{1-x^2}} - 2x^2\left(\Theta^{q-2}+\Xi^{q-2}\right)\right]\\ &+\Lambda(q-3)x^2\left[\left(\Theta^{q-1}-\Xi^{q-1}\right)^2-\left(\Theta^q+\Xi^q\right)\left(\Theta^{q-2}+\Xi^{q-2}\right)\right]\\ =& \Lambda\left(\Theta^q+\Xi^q\right)\left[\frac{\left(\Theta^{q-1}-\Xi^{q-1}\right)}{\sqrt{1-x^2}} - 2x^2\left(\Theta^{q-2}+\Xi^{q-2}\right)\right]\\ &-4\Lambda(q-3)x^{2q-2},\end{aligned} \tag{29}$$

where the equality is given by

$$\left(\Theta^{q-1}-\Xi^{q-1}\right)^2-\left(\Theta^q+\Xi^q\right)\left(\Theta^{q-2}+\Xi^{q-2}\right) = -4x^{2q-4}, \tag{30}$$

which is obtained from

$$\Theta+\Xi = 2,\ \Theta\Xi = x^2. \tag{31}$$

From the equality

$$\frac{\Theta^{q-1}-\Xi^{q-1}}{\sqrt{1-x^2}} = 2\left(\Theta^{q-2}+\Xi^{q-2}\right) + \frac{x^2\left(\Theta^{q-3}-\Xi^{q-3}\right)}{\sqrt{1-x^2}}, \tag{32}$$

Eq. (29) becomes

$$\begin{aligned}\frac{\mathrm{d}^2 g_{q,s}(x)}{\mathrm{d}x^2} \geq\ & \Lambda\left(\Theta^q+\Xi^q\right)\left[2(1-x^2)\left(\Theta^{q-2}+\Xi^{q-2}\right) + \frac{x^2\left(\Theta^{q-3}-\Xi^{q-3}\right)}{\sqrt{1-x^2}}\right] \\ & -4\Lambda(q-3)x^{2q-2}.\end{aligned} \tag{33}$$

Let us consider the binomial series for Θ^{q-1} and Ξ^{q-1},

$$\begin{aligned}\Theta^{q-1} &= \left(1+\sqrt{1-x^2}\right)^{q-1} \\ &= 1+(q-1)\sqrt{1-x^2} + \frac{(q-1)(q-2)}{2!}\left(\sqrt{1-x^2}\right)^2 + R_1\end{aligned} \tag{34}$$

and

$$\begin{aligned}\Xi^{q-1} &= \left(1-\sqrt{1-x^2}\right)^{q-1} \\ &= 1-(q-1)\sqrt{1-x^2} + \frac{(q-1)(q-2)}{2!}\left(\sqrt{1-x^2}\right)^2 + R_2,\end{aligned} \tag{35}$$

with remainder terms

$$\begin{aligned}R_1 &= \sum_{k=3}^{\infty}\frac{(q-1)\cdots(q-k)}{k!}\left(\sqrt{1-x^2}\right)^k, \\ R_2 &= \sum_{k=3}^{\infty}\frac{(q-1)\cdots(q-k)}{k!}\left(-\left(\sqrt{1-x^2}\right)\right)^k.\end{aligned} \tag{36}$$

Thus we have

$$\begin{aligned}&\Theta^{q-1}-\Xi^{q-1} = 2(q-1)\sqrt{1-x^2} + R_1 - R_2 \geq 2(q-1)\sqrt{1-x^2}, \\ &\Theta^{q-1}+\Xi^{q-1} = 2 + R_1 + R_2 \geq 2,\end{aligned} \tag{37}$$

for non-negative constants R_1-R_2 and R_1+R_2.

From inequality (33) together with (37), we have

$$\frac{\mathrm{d}^2 g_{q,s}(x)}{\mathrm{d}x^2} \geq 4\Lambda\left[2(1-x^2)+(q-3)(x^2-x^{2q-2})\right], \tag{38}$$

where the right-hand side of the inequality is always nonnegative for $0 \leq x \leq 1$ and $q \geq 1$. Thus $f_{q,s}(x)$ is monotonically increasing and convex on $0 \leq x \leq 1$ for $q \geq 1$, $0 \leq s \leq 1$ and $qs \leq 3$. □

We note that the monotonicity and convexity of $f_{q,s}(x)$ for $q \geq 1$ and $qs \leq 1$ are strict in the sense that the first and second derivatives of $f_{q,s}(x)$ are strictly positive for $0 < x < 1$. Now we prove Theorem 1, which relates concurrence with unified-(q, s) entanglement for two-qubit mixed states.

Proof of Theorem 1. For a two-qubit mixed state ρ_{AB} and its concurrence $\mathcal{C}(\rho_{AB})$, there exists an optimal decomposition of ρ_{AB}, in which every pure-state concurrence has the same value [8]; there exists a pure-state decomposition $\rho_{AB}=\sum_i p_i|\phi_i\rangle_{AB}\langle\phi_i|$ such that

$$\mathcal{C}(\rho_{AB})=\sum_i p_i\mathcal{C}(|\phi_i\rangle_{AB}), \tag{39}$$

and

$$\mathcal{C}(|\phi_i\rangle_{AB})=\mathcal{C}(\rho_{AB}), \tag{40}$$

for each i. Thus we have

$$\begin{aligned} f_{q,s}\left(\mathcal{C}(\rho_{AB})\right) &= f_{q,s}\left(\sum_i p_i\mathcal{C}(|\phi_i\rangle_{AB})\right)\\ &= \sum_i p_i f_{q,s}\left(\mathcal{C}(|\phi_i\rangle_{AB})\right)\\ &= \sum_i p_i E_{q,s}(|\phi_i\rangle_{AB})\\ &\geq E_{q,s}\left(\rho_{AB}\right). \end{aligned} \tag{41}$$

Conversely, the existence of the optimal decomposition of $\rho_{AB}=\sum_j q_j|\mu_j\rangle_{AB}\langle\mu_j|$ for unified-(q,s) entanglement leads us to

$$\begin{aligned} E_{q,s}\left(\rho_{AB}\right) &= \sum_j q_j E_{q,s}\left(|\mu_j\rangle_{AB}\right)\\ &= \sum_j q_j f_{q,s}\left(\mathcal{C}(|\mu_j\rangle_{AB})\right)\\ &\geq f_{q,s}\left(\sum_j q_j\mathcal{C}(|\mu_j\rangle_{AB})\right)\\ &\geq f_{q,s}\left(\mathcal{C}(\rho_{AB})\right), \end{aligned} \tag{42}$$

where the first and second inequalities are due to the convexity and monotonicity of $f_{q,s}(x)$ in Lemma 2. From Inequalities (41) and (42), we have

$$E_{q,s}\left(\rho_{AB}\right)=f_{q,s}\left(\mathcal{C}(\rho_{AB})\right) \tag{43}$$

for $q\geq 1$, $0\leq s\leq 1$, $qs\leq 3$ and any two-qubit mixed state ρ_{AB} $\square$

Due to the continuity of $f_{q,s}(x)$ with respect to q and s, we can always assure this functional relation between unified-(q,s) entanglement and concurrence in two-qubit systems for q slightly less than 1 or qs slightly larger than 3.

3. Multi-qubit monogamy of entanglement in terms of unified-(q, s) Entanglement

The monogamous property of a multi-qubit pure state $|\psi\rangle_{A_1A_2\cdots A_n}$ has been shown to be

$$\mathcal{C}^2_{A_1(A_2\cdots A_n)} \geq \mathcal{C}^2_{A_1A_2} + \cdots + \mathcal{C}^2_{A_1A_n}, \tag{44}$$

where $\mathcal{C}_{A_1(A_2\cdots A_n)} = \mathcal{C}(|\psi\rangle_{A_1(A_2\cdots A_n)})$ is the concurrence of $|\psi\rangle_{A_1A_2\cdots A_n}$ with respect to the bipartite cut between A_1 and the others, and $\mathcal{C}_{A_1A_i} = \mathcal{C}(\rho_{A_1A_i})$ is the concurrence of the reduced density matrix $\rho_{A_1A_i}$ for $i = 2, \ldots, n$ [4, 5]. Here, we show that this monogamy of multi-qubit entanglement can also be characterized in terms of unified-(q, s) entanglement. Before we prove multi-qubit monogamy relation of unified-(q, s) entanglement, we provide an important property of the function $f_{q,s}(x)$.

Lemma 3. *For $q \geq 2$, $0 \leq s \leq 1$ and $qs \leq 3$,*

$$h_{q,s}(x, y) := f_{q,s}\left(\sqrt{x^2 + y^2}\right) - f_{q,s}(x) - f_{q,s}(y) \geq 0, \tag{45}$$

on the domain $\mathcal{D} = \{(x, y)|0 \leq x, y, x^2 + y^2 \leq 1\}$.

Proof. In fact, Inequality (45) was already shown for $s = 0$ [22] and $s = 1$ [23]. Thus it is enough to consider the case that $0 < s < 1$.

As $h_{q,s}(x, y)$ is differentiable on domain $\mathcal{D}$, its maximum or minimum values arise only at the critical points or on the boundary of $\mathcal{D}$. By taking the first-order partial derivatives, we have the gradient of $h_{q,s}(x, y)$ as

$$\nabla h_{q,s}(x, y) = \left(\frac{\partial h_{q,s}(x, y)}{\partial x}, \frac{\partial h_{q,s}(x, y)}{\partial y}\right) \tag{46}$$

where

$$\begin{aligned}
\frac{\partial h_{q,s}(x, y)}{\partial x} =& \Gamma \frac{qsx\left[\left(1+\sqrt{1-x^2}\right)^q + \left(1-\sqrt{1-x^2}\right)^q\right]^{s-1}}{\sqrt{1-x^2}} \\
&\times \left[\left(1+\sqrt{1-x^2}\right)^{q-1} - \left(1-\sqrt{1-x^2}\right)^{q-1}\right] \\
&- \Gamma \frac{qsx\left[\left(1+\sqrt{1-x^2-y^2}\right)^q + \left(1-\sqrt{1-x^2-y^2}\right)^q\right]^{s-1}}{\sqrt{1-x^2-y^2}} \\
&\times \left[\left(1+\sqrt{1-x^2-y^2}\right)^{q-1} - \left(1-\sqrt{1-x^2-y^2}\right)^{q-1}\right], \\
\frac{\partial h_{q,s}(x, y)}{\partial y} =& \Gamma \frac{qsy\left[\left(1+\sqrt{1-y^2}\right)^q + \left(1-\sqrt{1-y^2}\right)^q\right]^{s-1}}{\sqrt{1-y^2}} \\
&\times \left[\left(1+\sqrt{1-y^2}\right)^{q-1} - \left(1-\sqrt{1-y^2}\right)^{q-1}\right]
\end{aligned}$$

$$
\begin{aligned}
&-\Gamma\frac{qsy\left[\left(1+\sqrt{1-x^2-y^2}\right)^q+\left(1-\sqrt{1-x^2-y^2}\right)^q\right]^{s-1}}{\sqrt{1-x^2-y^2}}\\
&\times\left[\left(1+\sqrt{1-x^2-y^2}\right)^{q-1}-\left(1-\sqrt{1-x^2-y^2}\right)^{q-1}\right] \qquad (47)
\end{aligned}
$$

with $\Gamma = 1/\left[(1-q)s2^{sq}\right]$.

Now, let us suppose there exists (x_0, y_0) in the interior of the domain $\mathcal{D}^\circ = \{(x,y)|0 < x,\ y,\ x^2+y^2 < 1\}$ such that $\nabla h_{q,s}(x_0,y_0) = (0,0)$. From Eq. (47), it is straightforward to verify that $\nabla h_{q,s}(x_0,y_0) = (0,0)$ implies

$$
n_{q,s}(x_0) = n_{q,s}(y_0), \qquad (48)
$$

for a differentiable function

$$
\begin{aligned}
n_{q,s}(x) :=& \frac{qs}{\sqrt{1-x^2}}\left[\left(1+\sqrt{1-x^2}\right)^q+\left(1-\sqrt{1-x^2}\right)^q\right]^{s-1}\\
&\times\left[\left(1+\sqrt{1-x^2}\right)^{q-1}-\left(1-\sqrt{1-x^2}\right)^{q-1}\right], \qquad (49)
\end{aligned}
$$

defined on $0 < x < 1$. Here we first show that $n_{q,s}(x)$ is a strictly increasing function for $0 < x < 1$, and thus Eq. (48) implies $x_0 = y_0$.

Let us consider the first derivative of $n_{q,s}(x)$. Because $xn_{q,s}(x) = \mathrm{d}g_{q,s}(x)/\mathrm{d}x$, where $\mathrm{d}g_{q,s}(x)/\mathrm{d}x$ is in Eq. (27), we have

$$
\frac{\mathrm{d}n_{q,s}(x)}{\mathrm{d}x} = \frac{1}{x}\left(\frac{\mathrm{d}^2 g_{q,s}(x)}{\mathrm{d}x^2} - n_{q,s}(x)\right), \qquad (50)
$$

for non-zero x. To show $n_{q,s}(x)$ is strictly increasing function, it is thus enough to show that $\mathrm{d}^2 g_{q,s}(x)/\mathrm{d}x^2 - n_{q,s}(x) > 0$ for $0 < x < 1$. By using Θ and Ξ (26), we have

$$
\begin{aligned}
\frac{\mathrm{d}^2 g_{q,s}(x)}{\mathrm{d}x^2} - n_{q,s}(x) =& \Omega\frac{x^2\left(\Theta^q+\Xi^q\right)\left(\Theta^{q-1}-\Xi^{q-1}\right)}{\sqrt{1-x^2}}\\
&+\Omega q(1-s)x^2\left(\Theta^{q-1}-\Xi^{q-1}\right)^2\\
&-\Omega(q-1)x^2\left(\Theta^q+\Xi^q\right)\left(\Theta^{q-2}+\Xi^{q-2}\right), \qquad (51)
\end{aligned}
$$

with $\Omega = qs\left(\Theta^q+\Xi^q\right)^{s-2}/\left(1-x^2\right)$.

Because $q(1-s) \geq q-3$ for $qs \leq 3$,

$$
\begin{aligned}
\frac{\mathrm{d}^2 g_{q,s}(x)}{\mathrm{d}x^2} - n_{q,s}(x) \geq& \Omega x^2\left(\Theta^q+\Xi^q\right)\left[\frac{\left(\Theta^{q-1}-\Xi^{q-1}\right)}{\sqrt{1-x^2}} - 2\left(\Theta^{q-2}+\Xi^{q-2}\right)\right]\\
&+\left[\left(\Theta^{q-1}-\Xi^{q-1}\right)^2-\left(\Theta^q+\Xi^q\right)\left(\Theta^{q-2}+\Xi^{q-2}\right)\right]\\
&\times\Omega(q-3)x^2. \qquad (52)
\end{aligned}
$$

Furthermore, from the relation $\Theta - \Xi = 2\sqrt{1-x^2}$, we have

$$
\frac{\Theta^{q-1}-\Xi^{q-1}}{\sqrt{1-x^2}} - 2\left(\Theta^{q-2}+\Xi^{q-2}\right) = \frac{x^2\left(\Theta^{q-3}-\Xi^{q-3}\right)}{\sqrt{1-x^2}}. \qquad (53)
$$

Together with Eq. (30) and Inequalities (37), we have

$$\frac{\mathrm{d}^2 g_{q,s}(x)}{\mathrm{d}x^2} - n_{q,s}(x) \geq 4\Omega(q-3)(x^4 - x^{2q-2}). \tag{54}$$

Here we note that the right-hand side of the inequality (54) is strictly positive for $0 < x < 1$ when $q \neq 3$; therefore, $n_{q,s}(x)$ is a strictly increasing function for $q \neq 3$ and $qs \leq 3$. For the case that $q = 3$, we have

$$n_{3,s}(x) = 12s\left(8 - 6x^2\right)^{s-1}, \tag{55}$$

which is also a strictly increasing function for $0 < s < 1$. In other words, $n_{q,s}(x)$ is a strictly increasing function for $q \geq 2$, $0 < s < 1$ and $qs \leq 3$, therefore Eq. (48) implies $x_0 = y_0$. However, from Eq. (47), $\partial h_{q,s}(x_0, x_0)/\partial x = 0$ also implies that $n_{q,s}(x_0) = n_{q,s}(\sqrt{2}x_0)$ for some $x_0 \in (0,1)$, which contradicts the strict monotonicity of $n_{q,s}(x)$; $n_{q,s}(x)$ has non-vanishing gradient in $\mathcal{D}^\circ$ for $q \geq 1$ and $qs \leq 3$.

Now let us consider the function value of $h_{q,s}(x,y)$ on the boundary of the domain $\partial\mathcal{D} = \{(x,y)|x = 0 \textit{ or } y = 0 \textit{ or } x^2 + y^2 = 1\}$. If either x or y is 0, then it is clear that $h_{q,s}(x,y) = 0$. For the case that $x^2 + y^2 = 1$, $h_{q,s}(x,y)$ is reduced to a single-variable function,

$$\begin{aligned} l_{q,s}(x) := &\frac{1}{(q-1)s2^{qs}}\left(\left(1+\sqrt{1-x^2}\right)^q + \left(1-\sqrt{1-x^2}\right)^q\right)^s \\ &+ \frac{1}{(q-1)s2^{qs}}\left[((1+x)^q + (1-x)^q)^s - 2^s - 2^{qs}\right]. \end{aligned} \tag{56}$$

In other words, the nonnegativity of $h_{q,s}(x,y)$ for $q \geq 2$ and $qs \leq 3$ follows from that of the differentiable function

$$\begin{aligned} m_{q,s}(x) := &\left(\left(1+\sqrt{1-x^2}\right)^q + \left(1-\sqrt{1-x^2}\right)^q\right)^s \\ &+ ((1+x)^q + (1-x)^q)^s - 2^s - 2^{qs}. \end{aligned} \tag{57}$$

From the derivative of $m_{q,s}(x)$,

$$\begin{aligned} \frac{\mathrm{d}m_{q,s}(x)}{\mathrm{d}x} = &sq\left[(1+x)^q + (1-x)^q\right]^{s-1}\left[(1+x)^{q-1} - (1-x)^{q-1}\right] \\ &- \frac{sqx}{\sqrt{1-x^2}}\left[\left(1+\sqrt{1-x^2}\right)^q + \left(1-\sqrt{1-x^2}\right)^q\right]^{s-1} \\ &\times\left[\left(1+\sqrt{1-x^2}\right)^{q-1} - \left(1-\sqrt{1-x^2}\right)^{q-1}\right], \end{aligned} \tag{58}$$

we note that $x = 1/\sqrt{2}$ is the only critical point of $m_{q,s}(x) = 0$ on $0 < x < 1$. Because $m_{q,s}(0) = m_{q,s}(1) = 0$ and $m_{q,s}(x)$ has only one critical point on $0 < x < 1$, $m_{q,s}(x)$ is either nonnegative or nonpositive through the whole range of $0 \leq x \leq 1$.

To show $m_{q,s}(x)$ is nonnegative on $0 \leq x \leq 1$, we show its nonnegativity for x near 1. Let us consider the derivative of $m_{q,s}(x)$ as x approaches 1. From Eq. (58) and Inequalities (37), we note that

$$\frac{\mathrm{d}m_{q,s}(x)}{\mathrm{d}x} \leq sq\left[(1+x)^q + (1-x)^q\right]^{s-1}\left[(1+x)^{q-1} - (1-x)^{q-1}\right]$$

$$- sqx2(q-1) \times \left[\left(1+\sqrt{1-x^2}\right)^q + \left(1-\sqrt{1-x^2}\right)^q\right]^{s-1} \tag{59}$$

for $0 \le x \le 1$; therefore

$$\lim_{x\to 1} \frac{\mathrm{d}m_{q,s}(x)}{\mathrm{d}x} \le sq\left[2^{qs-1} - (q-1)2^s\right]. \tag{60}$$

For $qs \le 3$ and $0 \le s \le 1$, 2^{qs-1} in the right-hand side of Inequality (60) is bounded above by 4, whereas $(q-1)2^s \ge q-1$. Thus, the right-hand side of Inequality (60) is always negative for $q > 5$. In other words, $m_{q,s}(x)$ is a decreasing function as x approaches to 1 with $m_{q,s}(1) = 0$, and thus $m_{q,s}(x)$ is a nonnegative function for $q > 5$.

For $q \le 5$, we consider the function value of a two-variable function $b(q,s) = 2^{qs-1} - (q-1)2^s$ on the compact domain $\mathcal{D}_2 = \{(q,s) | 2 \le q \le 5, 0 \le s \le 1, qs \le 3\}$. The first-order partial derivatives of $b(q,s)$ are

$$\frac{\partial b(q,s)}{\partial q} = 2^{qs-1} s \log 2 - 2^s, \tag{61}$$

and

$$\frac{\partial b(q,s)}{\partial s} = 2^{qs-1} q \log 2 - (q-1)2^s \log 2. \tag{62}$$

If we assume $b(q,s)$ has a critical point at (q_0, s_0) in the interior of the domain $\mathcal{D}_2^\circ = \{(q,s) | 2 < q < 5, 0 < s < 1, qs < 3\}$, Eq. (61) implies

$$\frac{\partial b(q_0,s_0)}{\partial q} = 2^{q_0 s_0 -1} s_0 \log 2 - 2^{s_0} = 0. \tag{63}$$

Furthermore, from Eq. (62) together with Eq. (63), we have

$$\begin{aligned}\frac{\partial b(q_0,s_0)}{\partial s} &= 2^{q_0 s_0 -1} q_0 \log 2 - (q_0-1)2^{s_0} \log 2 \\ &= \frac{2^{s_0}}{s_0}\left[q_0 - (q_0-1)s_0 \log 2\right].\end{aligned} \tag{64}$$

However $s_0 \log 2$ is strictly less than 1, therefore Eq. (64) is always nonzero in the interior of the domain. In other words, for any (q_0, s_0) in the interior of the domain, $\partial b(q_0,s_0)/\partial s$ is always nonzero conditioned $\partial b(q_0,s_0)/\partial q = 0$. Thus $b(q,s)$ has no vanishing gradient in the interior of the domain. Furthermore, it is also direct to verify that $b(q,s)$ is non-positive on the boundary of the domain, and thus $b(q,s)$ is non-positive for $2 \le q \le 5$, $0 < s < 1$ and $qs \le 3$.

Thus $m_{q,s}(x)$ is nonnegative on $0 \le x \le 1$ for $0 \le s \le 1$, $q \ge 2$ and $qs \le 3$, and this implies nonnegativity of $h_{q,s}(x,y)$ for the same domain of q and s. □

The following theorem yields a multi-qubit monogamy inequality in terms of unified-(q,s) entanglement.

Theorem 4. *For $q \ge 2$, $0 \le s \le 1$, $qs \le 3$ and a multi-qubit state $\rho_{A_1 \cdots A_n}$, we have*

$$E_{q,s}\left(\rho_{A_1(A_2\cdots A_n)}\right) \ge E_{q,s}(\rho_{A_1A_2}) + \cdots + E_{q,s}(\rho_{A_1A_n}) \tag{65}$$

where $E_{q,s}\left(\rho_{A_1(A_2\cdots A_n)}\right)$ *is the unified-*(q,s) *entanglement of* $\rho_{A_1(A_2\cdots A_n)}$ *with respect to the bipartite cut between* A_1 *and* $A_2\cdots A_n$*, and* $E_{q,s}(\rho_{A_1A_i})$ *is the unified-*(q,s) *entanglement of the reduced state* $\rho_{A_1A_i}$ *for* $i=2,\cdots,n$*.*

Proof. We first prove the theorem for n-qubit pure state $|\psi\rangle_{A_1\cdots A_n}$. Note Inequality Eq. (44) is equivalent to

$$\mathcal{C}_{A_1(A_2\cdots A_n)} \geq \sqrt{\mathcal{C}^2_{A_1A_2}+\cdots+\mathcal{C}^2_{A_1A_n}}, \tag{66}$$

for any n-qubit pure state $|\psi\rangle_{A_1(A_2\cdots A_n)}$. Thus, from Lemma 3 together with Eq. (66), we have

$$\begin{aligned}
E_{q,s}\left(|\psi\rangle_{A_1(A_2\cdots A_n)}\right) &= f_{q,s}\left(\mathcal{C}_{A_1(A_2\cdots A_n)}\right)\\
&\geq f_{q,s}\left(\sqrt{\mathcal{C}^2_{A_1A_2}+\cdots+\mathcal{C}^2_{A_1A_n}}\right)\\
&\geq f_{q,s}\left(\mathcal{C}_{A_1A_2}\right)+f_{q,s}\left(\sqrt{\mathcal{C}^2_{A_1A_3}+\cdots+\mathcal{C}^2_{A_1A_n}}\right)\\
&\qquad\vdots\\
&\geq f_{q,s}\left(\mathcal{C}_{A_1A_2}\right)+\cdots+f_{q,s}\left(\mathcal{C}_{A_1A_n}\right)\\
&= E_{q,s}\left(\rho_{A_1A_2}\right)+\cdots+E_{q,s}\left(\rho_{A_1A_n}\right),
\end{aligned} \tag{67}$$

where the first equality is by the functional relation between the concurrence and the unified-(q,s) entanglement for $2\otimes d$ pure states, the first inequality is by the monotonicity of $f_{q,s}(x)$, the other inequalities are by iterative use of Lemma 3, and the last equality is by Theorem 1.

For an n-qubit mixed state $\rho_{A_1(A_2\cdots A_n)}$, let $\rho_{A_1(A_2\cdots A_n)} = \sum_j p_j|\psi_j\rangle_{A_1(A_2\cdots A_n)}\langle\psi_j|$ be an optimal decomposition such that $E_{q,s}\left(\rho_{A_1(A_2\cdots A_n)}\right) = \sum_j p_j E_{q,s}\left(|\psi_j\rangle_{A_1(A_2\cdots A_n)}\right)$. Because each $|\psi_j\rangle_{A_1(A_2\cdots A_n)}$ in the decomposition is an n-qubit pure state, we have

$$\begin{aligned}
E_{q,s}\left(\rho_{A_1(A_2\cdots A_n)}\right) &= \sum_j p_j E_{q,s}\left(|\psi_j\rangle_{A_1(A_2\cdots A_n)}\right)\\
&\geq \sum_j p_j\left(E_{q,s}\left(\rho^j_{A_1A_2}\right)+\cdots+E_{q,s}\left(\rho^j_{A_1A_n}\right)\right)\\
&= \sum_j p_j E_{q,s}\left(\rho^j_{A_1A_2}\right)+\cdots+\sum_j p_j E_{q,s}\left(\rho^j_{A_1A_n}\right)\\
&\geq E_{q,s}\left(\rho_{A_1A_2}\right)+\cdots+E_{q,s}\left(\rho_{A_1A_n}\right),
\end{aligned} \tag{68}$$

where the last inequality is by definition of unified-(q,s) entanglement for each $\rho_{A_1A_i}$. □

Theorem 4 is a direct consequence of Lemma 3 when there is a functional relation between unified-(q,s) entanglement and concurrence in two-qubit systems. Here we note that Lemma 3 is also a necessary condition for multi-qubit monogamy inequality in terms of unified-(q,s) entanglement: for a three-qubit W-class state [24]

$$|\mathrm{W}\rangle_{ABC} = a|100\rangle_{ABC}+b|001\rangle_{ABC}+c|010\rangle_{ABC} \tag{69}$$

with $|a|^2+|b|^2+|c|^2=1$, it is straightforward to verify that

$$\mathcal{C}\left(|\mathrm{W}\rangle_{A(BC)}\right)=\sqrt{2|a|^2\left(|b|^2+|c|^2\right)},$$
$$\mathcal{C}\left(\rho_{AB}\right)=\sqrt{2|a|^2|b|^2},\ \mathcal{C}\left(\rho_{AC}\right)=\sqrt{2|a|^2|c|^2}, \tag{70}$$

where $\mathcal{C}\left(|\mathrm{W}\rangle_{A(BC)}\right)$ is the concurrence of $|\mathrm{W}\rangle_{ABC}$ with respect to the bipartite cut between A and BC, and $\mathcal{C}\left(\rho_{AB}\right)$ and $\mathcal{C}\left(\rho_{AC}\right)$ are the concurrences of the reduced density matrices $\rho_{AB}=\mathrm{tr}_C|\mathrm{W}\rangle_{ABC}\langle\mathrm{W}|$ and $\rho_{AC}=\mathrm{tr}_B|\mathrm{W}\rangle_{ABC}\langle\mathrm{W}|$ respectively. In other words, the CKW inequality (44) is saturated by $|\mathrm{W}\rangle_{ABC}$,

$$\mathcal{C}\left(|\mathrm{W}\rangle_{A(BC)}\right)^2=\mathcal{C}\left(\rho_{AB}\right)^2+\mathcal{C}\left(\rho_{AC}\right)^2. \tag{71}$$

Now suppose there is (x_0,y_0) in the domain $\mathcal{D}$ of the function $h_{q,s}(x,y)$ in Lemma 3 where the inequality (45) does not hold;

$$f_{q,s}\left(\sqrt{x_0^2+y_0^2}\right)-f_{q,s}(x_0)-f_{q,s}(y_0)<0. \tag{72}$$

In this case, we can always find a W-class state in Eq. (69) such that

$$x_0=\sqrt{2|a|^2|b|^2}=\mathcal{C}\left(\rho_{AB}\right),\ y_0=\sqrt{2|a|^2|c|^2}=\mathcal{C}\left(\rho_{AC}\right), \tag{73}$$

and thus

$$\begin{aligned}E_{q,s}\left(|\mathrm{W}\rangle_{A(BC)}\right)&=f_{q,s}\left(\mathcal{C}\left(|\mathrm{W}\rangle_{A(BC)}\right)\right)\\&=f_{q,s}\left(\sqrt{\mathcal{C}\left(\rho_{AB}\right)^2+\mathcal{C}\left(\rho_{AC}\right)^2}\right)\\&<f_{q,s}\left(\mathcal{C}\left(\rho_{AB}\right)\right)+f_{q,s}\left(\mathcal{C}\left(\rho_{AC}\right)\right)\\&=E_{q,s}\left(\rho_{AB}\right)+E_{q,s}\left(\rho_{AC}\right),\end{aligned} \tag{74}$$

which is a violation of the inequality in (65). Thus, Lemma 3 is a necessary and sufficient condition for multi-qubit monogamy inequality in terms of unified-(q,s) entanglement.

Although unified-(q,s) entanglement reduces to concurrence when $q=1/2$ and $s=2$, this case does not satisfy the condition of Theorem 4 for multi-qubit monogamy inequality. However, we note that the CKW inequality (44) characterizes the monogamy of multi-qubit entanglement in terms of squared concurrence rather than concurrence itself. In fact, Inequality (71) also implies that monogamy inequality of multi-qubit entanglement fails if we use concurrence rather than its square; for non-zero $\mathcal{C}\left(\rho_{AB}\right)$ and $\mathcal{C}\left(\rho_{AC}\right)$ in (71), we have

$$\mathcal{C}\left(|\mathrm{W}\rangle_{A(BC)}\right)^2=\mathcal{C}\left(\rho_{AB}\right)^2+\mathcal{C}\left(\rho_{AC}\right)^2\lneqq\left(\mathcal{C}\left(\rho_{AB}\right)+\mathcal{C}\left(\rho_{AC}\right)\right)^2, \tag{75}$$

and thus

$$\mathcal{C}\left(|\mathrm{W}\rangle_{A(BC)}\right)\lneqq\mathcal{C}\left(\rho_{AB}\right)+\mathcal{C}\left(\rho_{AC}\right). \tag{76}$$

Strictly speaking, concurrence does not show monogamy inequality of two-qubit entanglement whereas its square does in forms of CKW inequality.

For a bipartite pure state $|\psi\rangle_{AB}$, the squared concurrence is also referred as tangle

$$\tau\left(|\psi_{AB}\rangle\right) := \mathcal{C}\left(|\psi_{AB}\rangle\right)^2 = 2\left(1-\mathrm{tr}\rho_A^2\right), \tag{77}$$

and it is also extended to mixed states via the convex-roof extension,

$$\tau\left(\rho_{AB}\right) := \min \sum_i p_i \left(\mathcal{C}(|\psi_i\rangle_{AB})\right)^2 = \min \sum_i p_i \tau(|\psi_i\rangle_{AB}), \tag{78}$$

among all the pure state ensembles representing ρ_{AB} [4]. Thus tangle is always an upper bound of the squared concurrence for bipartite mixed state [25],

$$\begin{aligned}\tau\left(\rho_{AB}\right) &= \min \sum_i p_i \mathcal{C}(|\psi_i\rangle_{AB})^2 \\ &\geq \left(\min \sum_i p_i \left(\mathcal{C}(|\psi_i\rangle_{AB})\right)\right)^2 \\ &= \mathcal{C}(\rho_{AB})^2.\end{aligned} \tag{79}$$

In two-qubit systems, however, Eqs. (39) and (40) imply the existence of an optimal decomposition of ρ_{AB}, in which every pure-state concurrence has the same value, and thus Inequality (79) is always saturated in two-qubit systems;

$$\tau\left(\rho_{AB}\right) = \mathcal{C}(\rho_{AB})^2, \tag{80}$$

for any two-qubit state ρ_{AB}. In other words, the CKW inequality (44) can be rephrased as

$$\tau\left(\rho_{A_1(A_2\cdots A_n)}\right) \geq \tau\left(\rho_{A_1A_2}\right) + \cdots + \tau\left(\rho_{A_1A_n}\right), \tag{81}$$

for any n-qubit state $\rho_{A_1A_2\cdots A_n}$ [5].

Here we note that tangle is in fact a special case of unified-(q,s) entanglement. For $q=2$, $s=1$ and a bipartite pure state$|\psi\rangle_{AB}$, we have

$$E_{2,1}\left(|\psi\rangle_{AB}\right) = S_{2,1}\left(\rho_A\right) = 1-\mathrm{tr}\rho_A^2 = \frac{\tau\left(|\psi\rangle_{AB}\right)}{2}. \tag{82}$$

As both tangle and unified entanglement are extended to mixed states via the convex-roof extension, tangle can be considered as unified-$(2,1)$ entanglement up to a constant factor; therefore Inequality (81) is equivalent to the monogamy inequality in terms of unified-$(2,1)$ entanglement. In other words, Inequality (65) in Theorem 4 reduces to the CKW inequality when $q=2$, $s=1$.

We also note that Inequality (65) is reduced to the Rényi-q monogamy inequality [22]

$$\mathcal{R}_q\left(\rho_{A_1(A_2\cdots A_n)}\right) \geq \mathcal{R}_q(\rho_{A_1A_2}) + \cdots + \mathcal{R}_q(\rho_{A_1A_n}) \tag{83}$$

for $s\to 0$. For the case that $s\to 1$, Inequality (65) reduces to the Tsallis-q monogamy inequality [23]

$$\mathcal{T}_q\left(\rho_{A_1(A_2\cdots A_n)}\right) \geq \mathcal{T}_q(\rho_{A_1A_2}) + \cdots + \mathcal{T}_q(\rho_{A_1A_n}). \tag{84}$$

Thus, Theorem 4 provides an interpolation between Rényi and Tsallis monogamy inequalities as well as the CKW inequality, which is illustrated in Fig. 1.

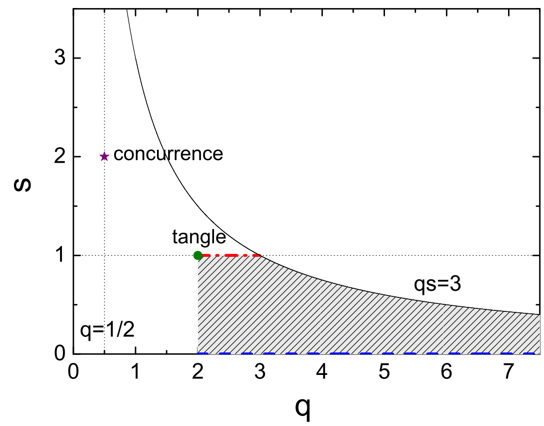

Figure 1. (Color online) The domain of q and s where multi-qubit monogamy inequality holds in terms of unified-(q, s) entanglement. The dashed line indicates the domain for which the multi-qubit monogamy inequality holds for Rényi-q entanglement, and the dashed-dot line is the domain for Tsallis-q entanglement. The shaded range is for unified-(q, s) entanglement.

We further note that the continuity of unified-(q, s) entropy also guarantees multi-qubit monogamy inequality in terms of unified-(q, s) entanglement when q and s are slightly outside of the proposed domain in Fig. 1.

4. Conclusion

Using unified-(q, s) entropy, we have provided a two-parameter class of bipartite entanglement measures, namely unified-(q, s) entanglement with an analytical formula in two-qubit systems for $q \geq 1$, $0 \leq s \leq 1$ and $qs \leq 3$. Based on this unified formalism of entropies, we have established a broad class of multi-qubit monogamy inequalities in terms of unified-(q, s) entanglement for $q \geq 2$, $0 \leq s \leq 1$ and $qs \leq 3$.

Our new class of monogamy inequalities reduces to every known case of multi-qubit monogamy inequality such as the CKW inequality, Rényi and Tsallis monogamy inequalities for selective choices of q and s. Our result also provides a necessary and sufficient condition for a multi-qubit monogamy inequality in terms of unified-(q, s) entanglement. Furthermore, the explicit relation between different monogamy inequalities was derived with respect to a smooth function $f_{q,s}(x)$. Thus, our result provides a useful methodology to understand the monogamous property of multi-party entanglement.

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[25] Because $\tau(\rho_{AB})$ is the minimum average of squared concurrence, the difference between $\tau(\rho_{AB})$ and $\mathcal{C}(\rho_{AB})^2$ is the *variance*, which is always nonnegative. Furthermore, the equality holds if every pure-state concurrence in the decomposition has the same value.